A GENERALIZATION OF LINDELÖF'S THEOREMS ON THE CATENARY.

BY PROFESSOR OSKAR BOLZA.

THE object of the following note is to generalize two well known theorems on the catenary due to Lindelöf* by proving the following proposition:

Whenever the general integral of Euler's differential equation for the integral

$$J = \int f(x, y, y')dx \tag{1}$$

is of the form

$$y = \alpha\varphi\left(\frac{x-\beta}{\alpha}\right) \tag{2}$$

(with α, β as constants of integration), the following two theorems hold:

A) If A and A' are a pair of conjugate points (in the wider sense) on an extremal for the integral (1), *then the tangents to the extremal at A and A' meet at a point T of the x-axis, and vice versa.*†

B) The value of the integral J taken along the arc AA' of the extremal is equal to the value of J taken along the broken line ATA':‡

$$J(AA') = J(AT) + J(TA'). \tag{3}$$

The proof of the first theorem is almost immediate. For if

$$\mathfrak{E}_0: \quad y = \alpha_0\varphi\left(\frac{x-\beta_0}{\alpha_0}\right) \tag{4}$$

be any particular extremal of the family (2) and $A(x_1, y_1)$ one of

*Compare Lindelöf-Moigno, "Leçons sur le calcul des variations," pp. 209–213.

†Compare my "Vorlesungen über Variationsrechnung," p. 80.

‡L. Bianchi has recently generalized Lindelöf's second theorem from the integral $\int y_1\sqrt{1+y'^2}dx$ to the more general integral $\int y^n\sqrt{1+y'^2}dx$, *Rendiconti della R. Accademia dei Lincei, Classe di scienze fisiche, matematiche e naturali*, series 5, vol. 19 (1910), p. 705. The extremals for this integral are of the form (2), so that Bianchi's result is contained as a special case in our theorem B).

its points, then the abscissa x_1' of the conjugate point A' satisfies the equation

$$u_1' - \frac{\varphi(u_1')}{\varphi'(u_1')} = u_1 - \frac{\varphi(u_1)}{\varphi'(u_1)}, \tag{5}$$

where

$$u_1 = \frac{x_1 - \beta_0}{\alpha_0}, \qquad u_1' = \frac{x_1' - \beta_0}{\alpha_0}.$$

On the other hand, the abscissa of the point of intersection T of the tangent to $\mathfrak{C}$ at A with the x-axis is

$$x_1 = \beta_0 + \alpha_0\left(u_1 - \frac{\varphi(u_1)}{\varphi'(u_1)}\right), \tag{6}$$

and the analogous quantity for the point A'

$$x_1' = \beta_0 + \alpha_0\left(u_1' - \frac{\varphi(u_1')}{\varphi'(u_1')}\right),$$

which, combined with (5), proves the first theorem.

In order to prove the second theorem we make use of a slight extension of Zermelo-Kneser's "envelop-theorem."*

Let

$$y = Y(x, a) \tag{7}$$

be a one-parameter set of extremals for the general integral

$$J = \int f(x, y, y')dx,$$

and suppose there exist two curves $\mathfrak{K}_1$ and $\mathfrak{K}_2$ both tangent to all the extremals of the set (4) (two branches of the envelop of the set (4)). If $\tilde{x}_1(a)$ and $\tilde{x}_2(a)$ are the abscissas of the points of contact 1 and 2 of the extremal (a) with the curves $\mathfrak{K}_1$ and $\mathfrak{K}_2$ respectively, these two curves may be written in parameter representation

$$\mathfrak{K}_1: \quad x = \tilde{x}_1(a),\ y = Y(\tilde{x}_1, a) \equiv \tilde{y}_1(a),$$

$$\mathfrak{K}_2: \quad x = \tilde{x}_2(a),\ y = Y(\tilde{x}_2, a) \equiv \tilde{y}_2(a),$$

and

$$Y'(\tilde{x}_1, a) = \frac{\tilde{y}_1'(a)}{\tilde{x}_1'(a)}, \qquad Y'(\tilde{x}_2, a) = \frac{\tilde{y}_2'(a)}{\tilde{x}_2'(a)}, \tag{8}$$

*Compare Kneser, *Mathematische Annalen*, vol. 50 (1898), p. 27.

whence it follows that

$$Y_a(\tilde{x}_1, a) = 0, \quad Y_a(\tilde{x}_2, a) = 0. \tag{9}$$

We now consider the integral J taken along the extremal (a) of the set (7) from the point 1 to the point 2. Its value is a function of a which we denote by $J(a)$, so that

$$J(a) = \int_{\tilde{x}_1}^{\tilde{x}_2} f(x, Y, Y')\, dx.$$

The derivative of this function can easily be computed by methods well known in the calculus of variations; if we take into account the equations (8) and (9), we obtain

$$J'(a) = f\left(\tilde{x}_2, \tilde{y}_2, \frac{\tilde{y}_2'}{\tilde{x}_2'}\right)\tilde{x}_2' - f\left(\tilde{x}_1, \tilde{y}_1, \frac{\tilde{y}_1'}{\tilde{x}_1'}\right)\tilde{x}_1'.$$

We integrate this equation with respect to a from a' to a'' and denote by $\mathfrak{E}'$ and $\mathfrak{E}''$ the extremals $a = a'$ and $a = a''$ of the set (7), by P', P'' the points of contact of $\mathfrak{K}_1$ with $\mathfrak{E}'$ and $\mathfrak{E}''$ respec-

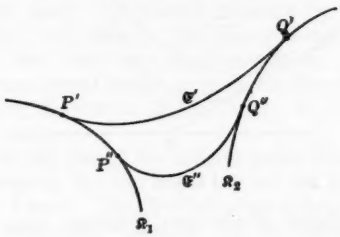

FIG. 1.

tively, by Q' and Q'' the points of contact of $\mathfrak{K}_2$ with $\mathfrak{E}'$ and $\mathfrak{E}''$ respectively. Then the result of the integration is

$$J_{\mathfrak{K}_1}(P'P'') + J_{\mathfrak{E}''}(P''Q'') = J_{\mathfrak{E}'}(P'Q') + J_{\mathfrak{K}_2}(Q'Q'').$$

But

$$J_{\mathfrak{K}_2}(Q'Q'') = -J_{\mathfrak{K}_2}(Q''Q').$$

Hence we obtain the desired extension of the envelop theorem in the form

$$J_{\mathfrak{E}'}(P'Q') = J_{\mathfrak{K}_1}(P'P'') + J_{\mathfrak{E}''}(P''Q'') + J_{\mathfrak{K}_2}(Q''Q'). \tag{10}$$

Our theorem B) can now easily be seen to be a special case of the general theorem just proved. For if the general integral of Euler's equation for the integral (1) is of the form (2), and if the tangents to the particular extremal (4) at A and A' meet at a point T of the x-axis, then the set of arcs directly homothetic with the arc AA' with respect to the center of similitude T are likewise extremals. For if c be the abscissa of the point T, these homothetic arcs are given by the equation

$$y = a\alpha_0\phi\left(\frac{x - (a\beta_0 + c(1-a))}{a\alpha_0}\right), \tag{11}$$

with a as ratio of similitude. They are therefore contained in the form (2) and are consequently extremals for the integral (1).

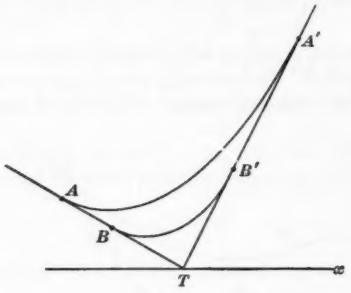

FIG. 2.

From the similitude of the arcs it follows that they are all touched at one of their extremities by the line TA, at the other by the line TA'. We may therefore apply to the set of extremals (11) and the two straight lines $\mathfrak{K}_1 = TA$ and $\mathfrak{K}_2 = TA'$ the above extension of the envelop theorem. Hence if BB' be one of the arcs in question for which $0 < a < 1$, then

$$J(AA') = J(AB) + J(BB') + J(B'A'),$$

and passing to the limit $a = 0$, we obtain the desired result

$$J(AA') = J(AT) + J(TA').$$

FREIBURG i. B.
July, 9, 1911.

A NOTE ON THE THEORY OF SUMMABLE INTEGRALS.

BY MR. S. CHAPMAN.

(Read before the American Mathematical Society, December 28, 1910.)

§ 1. *Introduction.*

Of recent years increasing use has been made of infinite series which do not converge; but whereas the mathematicians before the time of Abel and Cauchy used such series without proper examination of the validity of their use, modern mathematicians in general employ them only when the legitimacy of the work can be clearly demonstrated. The need so arising for theories to justify the application of ordinary methods and transformations to non-convergent series just as if they did converge has been fully recognized, and to meet it there have been published many well-known memoirs by Cesàro, Borel, Poincaré, LeRoy, and others. The theory is still being rapidly extended, and new important applications are constantly appearing. Among recent recent workers on the subject may be mentioned Bohr, Bromwich, Fejer, Hardy, C. N. Moore, and Riesz.

Parallel with the theory for infinite series there is a theory for infinite integrals, but the latter has not yet been developed so much as the former. Mr. Hardy* seems to have been the first to define a "summable" integral, and his paper was closely followed by one due to Dr. C. N. Moore,† in which some properties of summable integrals were proved. Subsequently Dr. Bromwich‡ wrote on the same subject, and more recently still the theory has been generalized by Mr. Hardy§ and

* *Quar. Jour. of Mathematics*, vol. 35, p. 54.

† "On the introduction of convergence factors into summable series and summable integrals," *Trans. Amer. Math. Society*, vol. 8, p. 299.

‡ "On the limits of certain infinite series and integrals," *Math. Annalen*, vol. 65, p. 350.

§ G. H. Hardy, "Notes on some points in the integral calculus," *Messenger of Mathematics*, vol. 40. "Theorems connected with Maclaurin's test for the convergence of series," *Proc. Lond. Math. Soc.* G. H. Hardy and S. Chapman, "A general view of the theory of summability of series and integrals," *Quar. Jour. of Mathematics*, 1911.

myself.* In particular I have applied methods of "summation" to series and integrals which *converge*, with the object, not, of course, of finding their sum, but of gaining an insight into the nature of their convergence.

§ 2. *Summable Integrals.*

Mr. Hardy's original definition of the summability of an integral was quite analogous to Cesàro's definition of the summability of a series by a first mean. Thus, if $f(x)$ is a function which is integrable in every finite interval lying in the interval $a \leqq x$, then the integral

$$\int_a^\infty f(x)dx$$

is said to be summable (C_1)† if the limit

$$\lim_{x \to \infty} \frac{1}{x} \int_a^x \int_a^\alpha f(\beta)d\beta d\alpha$$

exists; and the value of the limit is called the "sum" or "value" of the integral.

The extension of this definition is obvious, viz.,

The integral $\int_a^\infty f(x)dx$ is summable (C_r) if the limit

$$\lim_{x \to \infty} \frac{r!}{x^r} \int_a^x \int_a^{\alpha_1} \int_a^{\alpha_2} \cdots \int_a^{\alpha_r} f(\theta)d\theta d\alpha_r d\alpha_{r-1} \cdots d\alpha_1$$

exists; and as before the value of the limit defines the "sum" or "value" of the integral.

For many purposes, however, in the development of the general theory, the above multiple integral is more convenient when it is transformed into the expression

$$\int_a^x \left(1 - \frac{y}{x}\right)^r f(y)dy,$$

which is easily deduced by the theory of multiple integrals.

* S. Chapman, "On non-integral orders of summability of series and integrals," *Proc. Lond. Math. Soc.*, 1911.

† The convenient notation of which this is a particular case was introduced by Mr. Hardy in a paper published in the *Proc. Lond. Math. Soc.*, series 2, vol. 4, p. 257.

In particular the latter form is valuable because it shows the analogy of the above method of summation with that introduced (for the case of series) by Dr. M. Riesz.*

Further, it suggests the idea of non-integral orders of summability, and is an eminently suitable form for proving theorems on summability in full generality and with extreme simplicity.

Dr. C. N. Moore, in the paper already quoted, proved that if an integral is convergent, it is summable (C_1) and its "sum" is the same as the ordinary value. This is a particular case of the more general theorem which I have established,† to the effect that "if an integral is summable (C_r), then it is summable $(C_{r'})$, with the same sum, provided $r' > r$, where r and r' may be any positive or negative numbers (including zero), greater than -1." This may be called the general condition of consistency.

In this paper, however, I shall restrict myself to integrals which are summable (C_r), where r is zero or a positive integer only. I propose to discuss the behaviour of $f(x)$ as $x \to \infty$, in such integrals. The main result is an extension (with a certain redundant condition removed) of a theorem by Dr. C. N. Moore in his paper on convergence factors.‡

§ 3. *The Limit of the Integrand of a Summable Integral.*

In the case of an infinite series, it is a necessary but insufficient condition for convergence that $\lim_{n\to\infty} a_n = 0$. More generally, for a series $\sum a_n$ to be summable (C_r), it is necessary (though insufficient) that $\lim_{n\to\infty} (a_n/n^r) = 0$; this has long been known to hold when r is a positive integer,§ and I have proved that it holds generally for any value of $r > -1$.

We should not expect the same result to hold in the case

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* *Comptes Rendus*, July, 1909. Dr. Riesz's definition of the sum of a non-convergent series is

$$\lim_{n\to\infty} \sum_{v=0}^{n-1} u_v \left(1 - \frac{v}{n}\right),$$

and evidently n may be integral or not.

† "On non-integral orders . . . ," *Proc. Lond. Math. Soc.*, 1911.

‡ The method also is an extension of the one there used, but the question itself was presented to me otherwise, by the corresponding theorem for series.

§ See, for instance, Bromwich, The Theory of Infinite Series, Ch. XI.

of integrals,* for even in the case of convergent integrals (when $r = 0$) we know that $f(x)$ need not approach 0, and in fact may have no finite upper limit. An example of this will be given almost immediately.

It is well-known, however, that if $\int_0^\infty f(x)dx$ converges, then the condition of *uniform* continuity of $f(x)$, for $x \geqq k$, where k is any constant, is sufficient to ensure that $f(x) \to 0$ as $x \to \infty$.

Simple continuity of itself implies uniform continuity over any finite closed range of the variable, but this is evidently not the case over an infinite range.

Dr. Moore, who considered only integrals summable (C_1), proved that if $f(x)$ is uniformly continuous for $x \geqq k > 0$, and if $\int_0^\infty f(x)dx$ is summable (C_1), then

$$\lim_{x \to \infty} f(x)/x = 0.$$

This is a particular case of the more general theorem which I proceed to establish; but first it must be mentioned that the restriction upon $f(x)$ of uniform continuity is unnecessarily stringent. It is sufficient that $f(x)/x$ should be uniformly continuous for $x \geqq k > 0$. As Dr. Moore himself proved in the memoir referred to, if $f(x)$ is uniformly continuous for $x \geqq k > 0$, then so also is $f(x)/x^r$, for any $r > 0$; but the converse does not hold.

For example, consider the function $f(x)$ defined thus: $f(x) = 0$ for

$$n + 1/n^2 \leqq x \leqq (n+1) - 1/(n+1)^2;$$

* It may be interesting to point out that the method analogous to that which is successful in the case of series here fails completely. In proving the theorem for series, we use a formula for a_n in terms of $S_n{}^r$, viz.,

$$a_n = S_n^{(r)} - (r+1)S_{n-1}^{(r)} + \frac{(r+1)r}{1.2} S_{n-2}^{(r)} - \cdots$$

to r or $n+1$ terms according as r is or is not a positive integer.

The integral analogue of this expression is easily seen to be

$$\int_0^x (x-z)^{-(r+1)} \int_0^z f(t)(z-t)^r \, dt dz,$$

which is equal to

$$\int_0^x f(t)dt \int_t^x (z-t)^r (x-z)^{-(r+1)} \, dz = \int_0^x \frac{f(t)}{x-t} dt \int_0^1 \theta^r (1-\theta)^{-(r+1)} \, d\theta.$$

The second integral converges for no value of r, and therefore the method is useless.

and $f[n \pm (1-\theta)/n^2] = \theta\sqrt{n}$ for $0 < \theta < 1$. $\int_0^\infty f(x)dx$ is convergent but $f(n) = \sqrt{n}$, so that $f(x)$ does not $\to 0$ as $x \to \infty$, nor has it even a finite upper limit. $f(x)$ is continuous but not uniformly so, for $x \geqq k > 0$. But since $\int_0^\infty f(x)dx$ is summable (C_0), by the general condition of consistency it is summable (C_1); also $f(x)/x \to 0$, which however is not deducible from Dr. Moore's theorem, since $f(x)$ is not *uniformly* continuous. On the other hand, $f(x)/x$ *is* uniformly continuous, and hence $f(x)/x \to 0$ does follow from the theorem as we shall prove it, with the unnecessary restriction removed.

§ 4.

THEOREM. *If $f(x)/x^r$ is uniformly continuous for $x \geqq k > 0$, and $\int_0^\infty f(x)dx$ is summable (C_r), then*

$$\lim_{x \to \infty} f(x)/x^r = 0.$$

I restrict myself to the case when r is a positive integer. There seems little doubt, however, that the theorem holds for all values of $n > -1$.

Let $$L_r(x) = \int_a^x f(\alpha)\left(1 - \frac{\alpha}{x}\right)^r d\alpha.$$

By hypothesis, if L is the "sum" of the integral,

$$L_r(x) \to L \text{ as } x \to \infty.$$

Then, r being integral, we have

$$f(x) = \frac{1}{r!}\frac{d^{r+1}}{dx^{r+1}}\left[x^r L_r(x)\right], \tag{1}$$

which is immediately deducible from either form of $L_r(x)$. Since $f(x)/x^r$ is uniformly continuous for $x \geqq k > 0$, we can find a positive number δ, when any arbitrarily small positive constant ϵ is assigned, so that for $x \geqq k$,

$$\left|\frac{f(x+h)}{(x+h)^r} - \frac{f(x)}{x^r}\right| < \epsilon, \tag{2}$$

provided $0 < |h| < (r+1)\delta$.

But by the general mean value theorem of the differential calculus and equation (1) we have

$$(3)\quad \sum_{m=0}^{r+1}(-1)^m\binom{r+1}{m}(x+\overline{r+1-m\delta})^r L_r(x+\overline{r+1-m\delta})$$

$$= \delta^{r+1}\frac{d^{r+1}}{dx^{r+1}}[(x+\theta\delta)^r L_r(x+\theta\delta)]$$

$$= r!\,\delta^{r+1}f(x+\theta\delta)_r \qquad (0<\theta<r+1).$$

We now expand the expression (3) in the form of an integral polynomial in x. We get

$$x^r\sum_{m=0}^{r}(-1)^m\binom{r}{m}\{L_r(x+\overline{r+1-m\delta})-L_r(x+\overline{r-m\delta})\}$$
$$+x^{r-1}S,$$

where S contains only a finite number of terms (a number fixed when r is fixed) and, since $L_r(x)\to L$, has a finite upper limit K for all values of x and for all values of $\delta<1$, say. Further, we can find a constant x_0, depending on ϵ and δ, such that for $x'>x\geqq x_0$,

$$|L_r(x')-L_r(x)|<\epsilon\delta^{r+1}.$$

Hence for $x\geqq x_0$, we have

$$r!\,\delta^{r+1}f(x+\theta\delta)<2^r\cdot x^r\cdot\epsilon\cdot\delta^{r+1}+x^{r-1}S,$$

or

$$\frac{f(x+\theta\delta)}{x^r}<\frac{2^r}{r!}\epsilon+\frac{K}{r!\,\delta^{r+1}x}.$$

Hence if $x_0>1/\delta^{r+1}\epsilon$,

$$\frac{f(x+\theta\delta)}{(x+\theta\delta)^r}<K'\epsilon,$$

where K' is independent of x and δ. Now θ is a function of x, and hence $x+\theta\delta$ does not necessarily assume all values along the real axis. But by equation (2) it follows that for all values of $x\geqq x_0>1/\delta^{r+1}\epsilon$

$$\frac{f(x)}{x^r}<K''\epsilon,$$

where K'' is a positive constant, and consequently $f(x)/x^r \to 0$ as $x \to \infty$, which was what we desired to prove.

In conclusion, we may remark that the theorem may be stated as one of pure integral calculus, without reference to the theory of summability of integrals. Putting $f(x) = \phi(x)x^r$, the theorem thus becomes:

If $\phi(x)$ is uniformly continuous over the infinite interval $x \geqq k > 0$, then the convergence to a limit, as $x \to \infty$, of the integral

$$\int_0^x \phi(\beta)\beta^r \left(1 - \frac{\beta}{x}\right)^r d\beta$$

requires that $\phi(x)$ shall $\to 0$ as $x \to \infty$.

CAMBRIDGE,
ENGLAND.

IRREDUCIBLE HOMOGENEOUS LINEAR GROUPS OF ORDER p^m AND DEGREE p OR p^2.

BY PROFESSOR W. B. FITE.

(Read before the American Mathematical Society, February 25, 1911.)

No group all of whose non-invariant commutators give invariant commutators besides identity can be simply isomorphic with irreducible groups of different degrees. This category includes all groups of order p^m (p a prime) and classes one, two, and three. Moreover no group of order p^m can be simply isomorphic with irreducible groups of just two different degrees.*

A consideration of these facts gives rise to the query as to whether any group of order p^m can be simply isomorphic with irreducible groups of different degrees, and it is the purpose of this note to answer this question for certain special cases.

In the first place, if G is an irreducible group of order p^m and degree p, it cannot be simply isomorphic with an irreducible group of any other degree, since it contains an abelian subgroup of index † p, and since a group of order p^m with an abelian sub-

* BULLETIN, vol. 14 (1908), pp. 328, 329.

† *Transactions Amer. Math. Society*, vol. 7 (1906), p. 68. We shall have occasion to make use of the fact, established here, that in an irreducible group of order p^m and degree p, the substitutions commutative with a substitution that gives an invariant commutator besides identity form an abelian subgroup.

group of index p^s cannot be simply isomorphic with an irreducible group of degree greater than p^s.*

Suppose that G is an irreducible group of order p^m and degree p^s. If t is a substitution of G that corresponds to an invariant commutator of G' of order p, it is invariant in a subgroup G_1 of order p^{m-1} and G_1 must be reducible since t is invariant in it and is not a similarity substitution. We can so transform G as to exhibit G_1 in a completely reduced form with p irreducible components and with $G \equiv \{G_1, R\}$, where R replaces each variable of any component by the corresponding variable in the succeeding component.†

If G_1 contains a substitution s that has just p conjugates and is commutative with R, s must be the same in each of the components (each of degree p) of G_1. If s_i is that part of s that involves the variables of the ith component of G_1 then s_i is not invariant in its component, since if it were s would be invariant in G_1, and therefore invariant in G. Each component of G_1, as the ith one, being of degree p, contains an abelian subgroup of index p of which s_i is a part, and in the formation of G_1 no substitution of the abelian subgroup in one of these components can be associated with a substitution that is not in the corresponding abelian subgroup of some other component, since otherwise s would have more than p conjugates. Hence G_1 contains an abelian subgroup of index p, and G cannot be simply isomorphic with an irreducible group of any degree other than p^s.

Now t and R must be contained in a subgroup G_2 of order p^{m-1}, since otherwise G' would be of order p^2 and G could not be an irreducible group of degree p^2.‡ If then G_1 contains no substitution with just p conjugates that is commutative with R, G_2 can contain no invariant substitution that is not invariant in G. For every substitution of G_2 is of the form sR^α, where s is a substitution of G_1, and if sR^α is invariant in G_2, we must have $\alpha \equiv 0 \pmod{p}$, since s is commutative with t and R is not. Therefore sR^α is in G_1, and it is commutative with R. But G_1, by supposition, contains no substitution with just p conjugates that is commutative with R.

Hence G_2 is either irreducible or simply isomorphic with each

* Loc. cit., vol. 8 (1907), p. 110.

† But the variables in the last component are not necessarily replaced by the corresponding ones of the first component.

‡ Loc. cit., vol. 7 (1906), p. 67, Theorem II.

of its p irreducible components.* In the latter case, since these components are each of degree p, G_2 contains an abelian subgroup of order p^{m-2} and G cannot be simply isomorphic with an irreducible group of any degree except p^2. In the former case, since G contains an irreducible subgroup of order p^{m-1} and degree p^2, it cannot be simply isomorphic with an irreducible group of any degree except p^2 and p^3, if we assume that no group of order p^{m_1} $(m_1<m)$ can be simply isomorphic with irreducible groups of degrees p^2 and p^n respectively $(n>2)$. But no group of order p^m can be simply isomorphic with irreducible groups of just two different degrees. Hence G cannot be simply isomorphic with an irreducible group of any degree except p^2. The assumption upon which this conclusion rests can easily be justified for small values of m. We have therefore proved the following

THEOREM: *An irreducible group of order p^m and degree p^2 (p a prime) cannot be simply isomorphic with an irreducible group of any other degree.*

If we take into consideration the theorem just proved and the facts heretofore cited together with the fact that if G is simply isomorphic with an irreducible group of degree $p^{\frac{1}{2}(m-\alpha)}$, where p^α is the order of the central of G, it cannot be simply isomorphic with an irreducible group of any other degree,† we can readily verify that if a group of order p^m is simply isomorphic with irreducible groups of different degrees, then $m \geqq 12$.

Let G be an irreducible group of order p^m and degree p^2 that does not contain an abelian subgroup of index p^2. Then G_1 can contain no substitution that has just p conjugates and is commutative with R. Moreover the central of G' can contain only one subgroup of order p, since if t_1 and t_2 corresponded to independent operations of order p of the central of G', some substitution of the form $t_1 t_2^a$ would be commutative with R and would have just p conjugates in G_1. Hence the central of G' must be cyclic.‡

Any substitution of G that corresponds to an invariant operation of G' can give only invariant commutators; and if G' contains an invariant operation of order p^2, any corresponding substitution s of G has not more than p conjugates in G_1, since s^p is invariant in G_1. Moreover s is not invariant in G_1, since

* Loc. cit., vol. 8 (1907), p. 107.
† Ibid., p. 109; and BULLETIN, 2d series, vol. 14 (1908), p. 328.
‡ Burnside, Theory of Groups of Finite Order, pp. 73, 75.

it is not commutative with R^p. Hence in each component of G_1, as the ith one, s_i is contained in an abelian subgroup of index p, and the subgroup of G_1 within which s is invariant is abelian and of index p^2 under G.

We can assume then that the central of G' is of order p, and hence that the central of no succeeding cogredient contains an operation of order greater than p. If G were of class 3, G'' would be of order* p^2 and G' of order p^3. But this is impossible for an irreducible group of degree p^2.† We can assume then that G is of class k, where $k \geqq 4$.

Suppose that t_2 is a substitution of G that corresponds to an invariant operation of G''. Then it must be in G_1,‡ and if $t_{2,i}$ were invariant in the ith component of G_1, $t_{2,j}$ $(j = 1, 2, \ldots, p)$ would be invariant in the jth component, since in $R^{-1}t_2R$ the ith component is the same (except for the names of the variables) as the $(i-1)$th component of t_2, and since the commutator of R and t_2 is invariant in each component of G_1. Hence either t_2 is invariant in G_1, or $t_{2,i}$ is not invariant in the ith component and gives only invariant commutators. In the latter case the substitutions of the ith component of G_1 that are commutative with $t_{2,i}$ form an abelian group, and hence the substitutions of G_1 that are commutative with t_2 form an abelian group. Moreover this abelian group cannot be of index greater than p^2 under G_1, since t_2 cannot have more than p^2 conjugates under G_1. Hence G contains an abelian subgroup of index not greater than p^3.

We have now to consider the case in which t_2 is invariant in G_1. If under this supposition the central of G'' were of order greater than p, we could assume that every substitution of G that corresponded to an invariant operation of G'' is invariant in G_1. If then t_2 and s_2 were substitutions of G that corresponded to two independent operations of the central of G'', $t_2 s_2^{\alpha}$ would, for a suitably chosen value of α, give only invariant commutators in G. But this is impossible. Hence we consider the central of G'' to be of order p.

If now G were of class 4 $(k = 4)$, G''' would be generated by two independent generators each of order p, and hence G_1 would be abelian. But this is impossible in an irreducible group of degree p^2.

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*Loc. cit., vol. 3 (1902), p. 351.
†Loc. cit., vol. 7 (1906), p. 67.
‡Ibid., p. 62.

We now assume that an irreducible group of degree p^2 and order p^m contains an abelian subgroup of index less than p^k, if it is of class k, unless the central of each cogredient up to and including the $(k-j-1)$th is of order p, and the substitutions of G that correspond to any operation of these centrals are invariant in G_1. In this excluded case, if t_{k-j} is a substitution of G that corresponds to an invariant operation of $G^{(k-j)}$, it is contained in G_1. If it is not invariant in G_1, the substitutions of G_1 that are commutative with it form an abelian subgroup of index not greater than p^{k-j} and G contains an abelian subgroup of index less than p^k. If t_{k-j} is invariant in G_1, it can easily be shown (as in the case $k=4$) that we need only to consider the case in which the central of $G^{(k-j)}$ is of order p. Hence if our assumption holds for any value of $j(>1)$, it holds for the next smaller value of j. But we have shown that it holds for $j=k-2$.

We have assumed throughout that $k>2$. If $k=2$, G' is of order p^4 and contains an abelian subgroup of order p^{m-2}.*

We have proved therefore the

THEOREM: *If G is a group of order p^m and class k (>2) that is simply isomorphic with an irreducible group of degree p^2, it contains an abelian subgroup of index less than p^k. If $k=2$, G contains an abelian subgroup of index p^2.*

COLUMBIA UNIVERSITY,
August, 1911.

* Loc. cit., p. 67, and vol. 3 (1902), p. 343.

GRADUATE WORK IN MATHEMATICS IN UNIVERSITIES AND IN OTHER INSTITUTIONS OF LIKE GRADE IN THE UNITED STATES.*

1. *The Establishment of Advanced Instruction in the United States.*

FORTY years ago the bachelor's degree granted on the completion of a four years' course of a general character marked not merely the close of a young man's liberal education, but also, except in the case of some lawyers, ministers, and physicians, the end of all academic instruction of any kind. In particular, apart from a few exceptional cases, no advanced instruction in mathematics was anywhere provided beyond the usually rather meager ingredients—hardly more than analytic geometry and a little calculus—of this college course, which consisted mainly of prescribed studies. As an external sign of this state of affairs we note that the master's degree, where it existed, was conferred for reasons having very little to do with study, while the doctor's degree was practically non-existent.† The desire for higher education in America, which had been felt for many years by some of the leading minds of the country, had been able so far to achieve only momentary and sporadic success.

The most notable example of such a momentary success, so far as the study of mathematics is concerned, is to be found at Harvard during the fifties and early sixties, where, under the guidance of Benjamin Peirce, a band of young men devoted themselves successfully to the pursuit of higher mathematics.‡

* *International Commission on the Teaching of Mathematics.* Committee XII. Chairman, MAXIME BÔCHER, Harvard University; D. R. CURTISS, Northwestern University; P. F. SMITH, Yale University; E. B. VAN VLECK, the University of Wisconsin.

The subcommittee reports here referred to are:

1. Courses of Instruction and the Master's Degree. Chairman, Professor Curtiss. Cf. BULLETIN, February, 1911.

2. Preparation for Research and the Doctor's Degree. Chairman, Professor Smith. Cf. BULLETIN, March, 1911.

3. Preparation of Instructors for Colleges and Universities. Chairman, Professor Van Vleck. Cf. BULLETIN, November, 1910.

No attempt will here be made to summarize these reports completely or even to refer to all questions treated in them.

† Except at Yale University, where the degree of doctor of philosophy was established in 1860.

‡ Peirce was tutor or professor of mathematics at Harvard from 1831 till his death in 1880, but, except during the period here considered, it was only in the last ten years of his life that, under the influence of an expanding elective system, he again began to have an appreciable number of advanced students.

A few of these have since attained world-wide fame, while others were influential in introducing advanced mathematical instruction into the United States twenty or thirty years later. Peirce's success in collecting at this time a fair number of competent students for graduate work seems to have been due primarily to the presence of the office of the American Nautical Almanac at Cambridge from 1849 to 1866, and, secondarily, to the founding in 1847 of the Lawrence Scientific School,* which, in those early years, possessed, under the leadership of Louis Agassiz, Jeffries Wyman, Asa Gray, and others, some of the aspects of what is now known as a graduate school.

We notice, in passing, the contrast presented at this time, and for many years after, between the increasing supply of good astronomers in this country and the lack of men who, even by a stretch of the imagination, could be called mathematicians. It may fairly be said that the mathematical talent of the country was at this time diverted to astronomy.

Various circumstances united to bring a large measure of success in the establishment of graduate instruction in all fields, and in particular in mathematics, during the years 1870–1890. The great increase of wealth in the country brought with it endowments of many sorts which strengthened the older universities and established some important new seats of learning. Three things may be mentioned which, on this basis of material prosperity, did more than anything else to help forward the cause of graduate study in the critical period we are now considering.

1. *Study Abroad.*—For many years an occasional American had gone abroad to complete his studies. Thus B. A. Gould, a pupil of Benjamin Peirce and a graduate of the class of 1844 at Harvard, who later became eminent as an astronomer, studied with Gauss in Göttingen and took his doctor's degree there in 1848. Similarly J. Willard Gibbs after taking his doctor's degree at Yale in 1863 spent three years (1866–1869) in Paris, Berlin, and Heidelberg, where he studied with Kirchhoff, Helmholtz, Weierstrass, and others. A few more cases of a similar sort might be recorded, but it was not until the end of the

* In the same year the department of Philosophy and the Arts was organized at Yale with the purpose of furnishing "resident graduates and others with the opportunity of devoting themselves to special branches of study," these branches embracing "theology, law, medicine, and more particularly mathematical science and physical science and its applications." It was in this department that the doctor's degree was established in 1860, as noted above.

seventies or the beginning of the eighties that the stream of mathematical students from America to Europe (generally to Germany) became a steady one. This tendency to go to Germany for the closing years of study contributed probably more than anything else to build up sound standards of productive scholarship and of graduate teaching, without which all attempts to establish advanced instruction in this country must have remained abortive. Its success was in part due to the establishment and the wise administration of traveling fellowships, first at Harvard and then, to a much less degree, elsewhere. We shall return to this important matter of study abroad in a later section.

2. *The Foundation of Johns Hopkins University.*—The magnificent bequest of Johns Hopkins of three and a half million dollars for the foundation of a university in Baltimore, and his wisdom in leaving his board of trustees a free hand in the organization of the institution resulted in the adoption, on President Gilman's initiative, of a plan whereby the ordinary undergraduate instruction was relegated to a subordinate position from the very start, so that the new university stood before the American public as the standard bearer of the higher education. This was of inestimable benefit in strengthening the hands of those members of the faculties of the older universities who had been struggling to establish and develop graduate instruction at their own institutions. The presence of the eminent English mathematician, Sylvester, as professor of mathematics during the first seven years of the Johns Hopkins University, had also a marked effect in stimulating interest in advanced mathematical studies in America, though it is easy to overestimate his direct influence, as he was a poor teacher with an imperfect knowledge of mathematical literature. He possessed, however, an extraordinary personality, and had in remarkable degree the gift of imparting enthusiasm, a quality of no small value in pioneer days such as these were with us.

3. *The Elective System.*—At the beginning of the period under consideration the lack of students qualified to undertake advanced work was most keenly felt and made any large success in the establishment of graduate instruction an impossibility. The adoption under the lead of President Eliot, first at Harvard and then to a greater or less extent throughout the country, of a far-reaching elective system in the four-years' undergraduate course furnished a possibility for the gradual extension of

instruction in the special fields. Without entering on the question of the advantages and disadvantages of the elective system for the college itself, we may safely say that it provided a basis for advanced instruction without which any considerable development of such instruction, at least during the years of which we are now speaking, would hardly be conceivable.*

At the close of the period we are considering, when the idea of graduate instruction had already taken a firm hold on many of the stronger institutions of the country, the founding of Clark University exclusively for graduate study in mathematics, psychology, biology, physics, and chemistry gave a further impetus to specialization in advanced work, and the opening of the University of Chicago in 1892 may almost be said to mark an epoch in the development of graduate instruction in the West and Middle West; for, though that university had from the start an undergraduate department, it stood out, through the character of its faculty and the emphasis laid on research work, as a strong exponent of the graduate idea.

While in these universities, as well as at Johns Hopkins, advanced instruction was at once placed in a department by itself, in by far the larger number of institutions it developed very gradually within the old college, room being made for it by the elective system; and it was only slowly, even in the larger institutions, that small groups of graduate students began to collect. The somewhat unorganized condition, which was then the rule, is still to be found in the weaker institutions of the country and also in some colleges which in their chosen field of undergraduate work are strong, but which voluntarily renounce any substantial development of graduate instruction. The great universities, however, have all, since the year 1890, developed well-organized graduate schools frequented by the graduates of their own and other colleges. It may be added, to avoid possible misconception, that the graduate schools which sprang full-fledged into existence and those which developed slowly from the old college no longer form two distinct

* Cf., however, the closing remarks of § 2. What we desire to emphasize here is that an elective system so arranged as to allow some specialization in individual departments, not merely the choice between various *elementary* subjects, permitted a gradual development of more and more advanced instruction, the students being at first mainly undergraduates. *Such* a development could go on simultaneously at many places, while even a single attempt to duplicate the Johns Hopkins experiment would probably have quickly led to disastrous failure.

classes. Some of the strongest graduate schools in the country are now to be found among the last-named institutions.

In contrast with such countries as France, Italy, and, to a less extent, Germany, we note the complete lack of central control or organization in the United States. Many variations are hereby made possible which are, for a country like ours, almost a necessity; and competition, on the whole healthy, springs up between the different institutions.

In conclusion we note that of late years some technological schools (for instance the Massachusetts Institute of Technology) have undertaken a limited amount of graduate instruction in mathematics. As this instruction does not differ, except in the greater emphasis laid on applied mathematics, from that given at colleges and universities, and since the amount of such instruction at technological institutions is as yet very small in comparison to the whole amount of mathematical graduate instruction in the country, we have not thought it necessary to mention these technological schools specifically in what follows.

2. *The Graduate Student of Mathematics at the Present Day.*

Owing to the great variety of standards for the bachelor's degree in the different colleges of the country, the students of a single graduate school enter it with very diverse preparation. This is, however, not so disturbing as might be expected, owing to the fact that at every university in which a graduate school exists there is a collegiate or undergraduate department whose instruction is freely open to the graduate student who is in need of it. We may then say that not all work done by graduate students is graduate work. On the other hand, the ambitious and capable senior in colleges allowing considerable freedom of election will frequently be doing work of a distinctly graduate character in the same classes with able graduates of colleges in good standing.

If we thus miss any sharp line of demarcation at the lower limit of the graduate school between graduates and undergraduates, we find a similar phenomenon at the upper limit where the graduate student often passes by almost imperceptible steps into the teacher. Indeed there are graduate schools, even among the better institutions of the country, the bulk of whose students are at the same time assistants or instructors. This, and the very high percentage of graduate students of

mathematics the country over who are fellowship and scholarship holders are features of American education which it is to be hoped, will gradually pass away.* They are closely related to the presence in graduate schools of large numbers of students of mathematics who have reached an age when their student days should be over. It can not be too strongly urged on all who give young men advice or who are influential, by awarding scholarships or otherwise, in shaping their careers that it is only in his first youth (not at the age of thirty or thirty-five) that the foundation of real success can be laid by the student of mathematics.

A somewhat different class is formed by school-teachers in active service who are at the same time enrolled as graduate students of mathematics, but at any moment take necessarily only a small amount of work. The ambition of these teachers to improve their professional equipment is most laudable. When, however, as is sometimes the case, they form a considerable proportion of the enrollment of a graduate school, they may be a source of weakness to that school in spite of their earnestness of purpose.

The period spent by a student in graduate study varies from one to three, or even more years; and the amount of migration from one university to another does not seem to be large, although the great majority of students attend a graduate school at a different university from that at which their undergraduate years were spent.

We note also that in graduate work coeducation is the almost universal rule, not only in the great state and other western institutions where coeducation forms an integral part of the scheme of education from top to bottom, but even in the most conservative institutions of the East, which do not admit women to their undergraduate departments. Apart from Princeton and the University of Virginia, where no women are admitted, it is only in women's colleges (Bryn Mawr, Vassar, etc.) and in some institutions for men which have held firmly to the undergraduate idea, so that the amount of graduate work is very limited, that one sex alone will be found.

A striking and significant fact is that nearly half of all graduate students of mathematics come from small colleges. This is

* Nothing in any way resembling this free award of financial aid is found necessary to induce strong men to attend schools of law or engineering. Cf. the closing lines of this section.

probably due to the fact that in such colleges students always have the opportunity to study the elements of mathematics and often something beyond the elements, while the inducements for them to turn away into other fields are slight in comparison to those offered at larger institutions where a richer elective system prevails. The tendency so strong in our day and country to regard the man of action as being of nobler clay than the man of thought and ideas, reinforced by the much greater financial prizes open to the former, whether he be lawyer, business man, or engineer, creates a situation where it is not easy to secure for mathematical study a due proportion of the strongest youth in our college communities.

3. *The Organization of Advanced Mathematical Instruction.**

The purpose of mathematical instruction should be fourfold:

I. To impart knowledge.

II. To develop power and individual initiative.

III. To lead the student to express adequately and clearly what he knows.

IV. To awaken the love of knowledge and to impart scholarly ideals.

The first of these aims, without attention to which the other three can not be obtained, has led to the great development of the lecture system which we find in all graduate schools, and to the use of the textbook and treatise either in connection with these lecture courses or independently of them, and of the original memoir the reading of which constitutes an art by no means easy to acquire, and which deserves special cultivation at the hands of the members of the teaching staff.

As means used under II may be mentioned: first, the solution of problems by students either in connection with the lecture courses or in special seminars or proseminars, and secondly, the writing of theses which may or may not be connected with the doctor's degree. This last is also the chief means employed under III, although the quiz (cf. Subcommittee 3, § 4) is sometimes employed effectively for this purpose, and even the brief written problem is not without some value here.

Both II and III above should receive more attention than is now commonly given to them, while I is at present adequately treated, except, perhaps in the relative weakness of applied as distinguished from pure mathematics.

* Cf. also the report of Subcommittee 1.

The aim indicated under IV depends for its attainment less on special methods of instruction and more on the personality of the instructor and his attitude towards science than do I, II, or III. To secure adequately the end in view, an instructor is needed who combines high *scientific* ideals with a commanding or sympathetic personality. Such men could do much to counteract the tendency noted in the closing lines of § 2, but, on the other hand, it is precisely this tendency which makes them difficult to secure. Their influence on undergraduate instruction should be no less valuable than in the graduate school.

We must come back once more to the lecture courses which everywhere form the backbone of graduate mathematical instruction. Such a course usually extends either through the whole academic year, that is from the end of September till early in June, or through the first or second half of this year.* The lectures, of somewhat less than an hour each, come usually three times (less frequently twice) a week. While much depends on the local traditions and the personality of the instructor, it may in a general way be said that these lectures have a far less formal character than is the case in European universities. Students will frequently interrupt the lecturer with a question, and short discussions between the instructor and one or more students will not infrequently take place, and at times the more formal quiz (cf. Subcommittee 1, § 6 and Subcommittee 3, § 4) finds its place here. Some tact and firmness are occasionally necessary to prevent the loquacious or thick-headed student from monopolizing the time of the class, but on the whole this frequent contact during the lecture between teacher and student is an admirable feature of American higher education, and counteracts, to a certain extent, some evils which usually accompany the lecture system. It is made possible by the smallness of the classes, an audience of twenty-five in a graduate course in mathematics being distinctly unusual.

The range of subjects covered by the courses offered in each graduate school is very great (cf. Subcommittee 1, § 5). This is peculiarly the case in those institutions which have only recently begun a policy of expansion in their graduate work, where the first sign of such expansion often appears in an astounding increase in the number and range of courses offered, for only a small part of which there are students. Indeed, if

* Attention must also be called to work in the summer schools and summer quarters. Cf. Subcommittee 1, § 4.

students should present themselves, the capacity of the teaching force would be completely overtaxed. This is a state of affairs which no self-respecting institution should allow to continue, and there are signs that it is usually of only a temporary nature, since with a real strengthening of the mathematical department of such an institution this inflation tends to disappear. We hasten to add that the stronger institutions, and many smaller institutions with a due sense of proportion, offer admirable selections of courses commensurate with their capacity and the needs of their students, courses which at each institution usually vary considerably from year to year. Even in the weaker institutions where a call for advanced instruction is hardly apparent, it may often be wise to encourage instructors to offer a course of a not wholly elementary character, as it will frequently be found to act as a tonic and, by keeping them in touch with the scientific side of their subject, enable them to make their elementary work more vital.

It was mentioned in § 2 that no sharp distinction between graduate and undergraduate work in mathematics can be made. Indeed it is hard to exclude entirely from graduate work anything above the first course in the calculus, now commonly taken in the second undergraduate year. The actual state of affairs is best expressed by regarding the group of courses just following this point, such as a second course in the calculus, the elements of determinants and of the theory of equations, projective geometry, a first course on differential equations, etc., as belonging both to graduate and to undergraduate instruction. From this latter point of view, however, these courses usually appeal only to the student of distinct mathematical ability and seriousness of purpose, whose presence in the course along with graduates does not very greatly affect the character of the course.

As the external signs of success for the graduate student we have the master's and the doctor's degrees. The first of these is commonly given for one year's graduate work done largely in one subject, such as mathematics or physics, and tested either by course examinations in which a higher standard is demanded than is accepted for undergraduates, or by a single examination covering the whole year's work. A thesis is also often required for the master's degree; but the work done on this thesis is not commonly of the nature of research work, and the degree is taken by considerable numbers of students most of whom never

proceed further. This degree is given, and properly given, by a large number of institutions, many of which have only a very moderate strength in their graduate mathematical work. Under these conditions suggestions for a minimum standard for the degree are not out of place, and such suggestions will be found in the report of Subcommittee 1, § 7.

The doctor's degree originally came to us from Germany, but has long been naturalized and is in all American institutions of good standing distinctly a research degree. In several of our stronger universities it has a standard at least as high as the best German standard. The requirements for the doctor's degree in universities which have been giving it to any extent during the last ten years are tolerably uniform (cf. the report of Subcommittee 2), but in this matter so much depends on the unwritten standards of individual professors or departments that there still remains a great difference in the ease with which the degree can be obtained at different institutions. It is for this reason that the suggestion which is sometimes made that it would be well to attempt to formulate definite standards for the doctor's degree to which the universities of the country should conform, seems to be of slight practical value.

In school and college work America adopts in one respect a very different standpoint from France and Germany, and this has a certain indirect influence on graduate work. We refer here to the fact that in the last-named countries a pupil will not be allowed to proceed from class to class, and, particularly, will not be allowed to pass the great educational landmarks (for instance graduation from the gymnasium in Germany) without conforming to a very exacting standard which a considerable percentage of each class fails to attain. In America, on the other hand, the teacher who tries to impede seriously the progress of any but the unusually lazy or stupid soon makes himself impossible. This is not the place to discuss the respective merits of these two points of view in the secondary school or even in the college; but when we come to the graduate student of mathematics it seems clear that the American attitude must be modified, and, as a matter of fact, in all the stronger institutions of the country a much greater ability and earnestness of purpose is demanded for passing examinations and securing degrees in the graduate school than would be allowed to pass muster in undergraduate work. Nevertheless, it is to be hoped that something more will be accomplished in this direction, and

that, in particular, candidates for the doctor's degree will be made to feel that success for them at an institution of good standing is not a mere matter of time and patience. It is the more important to insist on this since, as has just been said, the whole current of secondary and college education runs in another channel.

4. *Teachers.*

We must be concerned with this subject for two different reasons: first, because the great majority of graduate students of mathematics ultimately become teachers in secondary schools, colleges, or universities; and, secondly, because on the quality and efficiency of the teachers in the graduate school itself (professors, instructors, etc.) depends to such a large degree the quality of the school.

It is a favorable sign of the gradual elevation of the profession of secondary-school teacher that of late years many persons wishing to adopt this profession spend a year in study in a graduate school. It is true that this time is frequently not spent in the study of a single subject; but for the future teacher of mathematics (or of mathematics and some other subject) to have had a couple of graduate courses in mathematics, usually in the intermediate group referred to on p. 130, is a very substantial gain over the conditions of twenty years ago. It is to this class of students that the courses on the teaching of mathematics, which are now given at many colleges and universities, mainly appeal.

If we except this group who go into secondary-school teaching, and a second group who study mathematics as a tool for use in some other science, such as physics, it may be said with almost absolute precision that all other students of mathematics in graduate schools become instructors in mathematics in colleges or universities. The condition of twenty-five years ago, where college instructors in mathematics were taken from among the freshly graduated students of a college (usually the same college where they were to teach), has now become the exception instead of the rule; and where it still occurs, the appointment is usually a temporary one, both the instructor and the college expecting that, after a year or two of teaching, further graduate study will follow. The gain involved in this changed state of affairs, both in breadth of view and in real mastery of the subject, the

teaching of which is to be the young man's life work, is so obvious as to require no further comment here. If the student can furthermore be given some comprehension of the fact that the science of mathematics is a living and growing one through contact with other students or instructors who are themselves contributing to this growth, and still more if he himself can take some part in the development of mathematical knowledge, his outlook on mathematics in particular and intellectual life in general will have been so broadened that he can hardly fail to become a better member of a college faculty than would otherwise have been the case.

After all this has been said, we must, however, admit that this question has also another side less pleasant to contemplate. What passes for original research, in this country more even than abroad, is often hardly a real contribution to mathematical progress at all, but merely a grinding out of results, which if they have only never been published before may be as unimportant and unattractive as you please; they form an "original contribution." One is tempted to answer: Yes, in the same sense as the brass button in the contribution box. We may feel certain that in the long run this will be the character of the research work done by students who have no real capacity or inclination for original work, but who are pushed into it by the increasing demand, on the part of certain heads of departments, for the doctor's degree as a necessary preliminary to college teaching. The pressure thus produced will surely, if persisted in, bring forth an increasing yearly crop of doctors,—success can be obtained by almost anyone with a fair mathematical capacity and with sufficient industry and patience, either by going abroad or by going to one of the weaker American institutions with an ambition for giving the doctor's degree. It is doubtful if the time will ever come, certainly it will not come for a great many years, when all the members of the teaching staffs of the large universities of the country, and the colleges of like rank, can be men with a real capacity for original investigation; the number of all such men in the country falls far short of (one might almost say that it is of a different order of magnitude from) the number of places to be filled.

The pseudo doctor, to whom reference was made above, is often narrowed rather than broadened by the bit of investigation which he has been set to do, and becomes thereby less effective as a teacher, investigation for him becoming a fetich

for which he forgets all other ideals. Or, on the other hand, he may let all thought of original work drop out of his mind when once he has secured his degree. In either case the letters he places after his name ought not to go very far in recommending him for teaching positions. A broad and deep mathematical training should surely be demanded by all the institutions of the country which claim collegiate rank as a prerequisite for a permanent appointment on their teaching staff. They will naturally demand also some ability as a teacher. If in addition they can secure an investigator of a genuine sort, even though his caliber be slight, they should usually regard themselves as fortunate, though a few of the strongest institutions can and should set themselves a much higher standard. On the other hand, our stronger graduate schools should continue, as they are now doing, to encourage every capable student to try his hand at some piece of original investigation, but they should not hesitate, after a fair trial, to tell him, if that turns out to be the case, that he is not fitted for that kind of work.

No specific training for the profession of college or university instructor is commonly given in graduate schools apart from the training in mathematics (cf. Subcommittee 3, § 5). The statement made in § 3 of the present report that the training in clear and adequate exposition which is given to graduate students of mathematics is frequently insufficient is of peculiar importance in relation to the future teacher. While it is probably not desirable to attempt to train the future college or university instructor in the art of teaching, the question whether more can not be done to lead graduate students of mathematics to express their ideas well both in spoken and in written form is worthy of serious consideration.

Let us turn now from the graduate students, who are to become college instructors, to the actual instructors and professors of mathematics in our colleges and universities. If we compare conditions at the present day with those existing twenty years ago, a very great increase in the standard of mathematical knowledge on the part of the teaching staff is evident. That the improvement here has not been even greater is due in large measure to the fact that the supply of well-trained graduate students falls far short of the demand. Weak appointments are also made from time to time, owing to ignorance on the part of trustees or heads of departments of what really constitutes a mathematician, to the pernicious view that administrative

ability may be allowed to take the place of mathematical ability, or to other like causes. Flagrant cases of this kind occasionally occur which make one blush for the good name of American universities, but such cases are now merely sporadic and one gains comfort by contemplating conditions in Germany only a hundred years ago. What is needed here, as in so many other places in American life, is a strengthening of *intelligent* idealism (we have more than enough misdirected idealism amongst us) based upon knowledge, and there seems every reason to hope that the great development of mathematics in this country during the last twenty years, evident chiefly in the growth and activity of the American Mathematical Society, will in an ever-increasing degree supply the intelligent and influential public opinion here needed. The shortage, above mentioned, in the supply of instructors in mathematics forms the most serious aspect of the situation.

For various further points: The excessive burdening of young instructors with drudgery, which still often occurs; inadequate salaries; the burdening of professors with administrative work; we refer to the report of Subcommittee 3.

5. *Study by Americans Abroad.*

No account of higher mathematical education in America would be complete without a reference to the part played by the study of Americans abroad. What an important factor this was in introducing advanced mathematical instruction and research into America has already been mentioned in § 1. In the early days the possibilities for advanced mathematical study in this country were very limited, so that it was natural that students able to do so should go abroad where they could find this opportunity in large measure. At the present day it may safely be said that at several of the stronger American graduate schools most American students find mathematical opportunities better suited to their needs than are to be found at any place abroad. Nevertheless, students still go abroad in apparently undiminished numbers to study mathematics,* and

* It would be a matter of considerable interest to have statistics on the number of American students who go abroad each year to study mathematics and the length of time they stay. Such statistics would seem to be very difficult if not impossible to secure. As to the proportion of instructors of graduate courses in mathematics who have spent at least one year abroad, see the report of Subcommittee 1, § 3. Far less important is the question of the number of doctor's degrees conferred on Americans

their decision to do this is frequently a wise one. Let us inquire how this can be the case.

There come first considerations of a non-mathematical character. It is desirable for everyone to become acquainted at first hand with other countries than his own, and this is doubly true for an American, for whom a period of residence in European countries is invaluable. It is true that the student often seems to have brought back from a year or two of residence abroad only a strengthening of his earlier national prejudices, since the mote in the neighbor's eye is so very easy to discern;* but if he is worth his salt, he brings with him a fund of impressions and experiences which, as time goes on, greatly enrich his life. For this reason alone study abroad is to be recommended even at some mathematical sacrifice. A second consideration is that the cost of living in Germany, to which country the great majority of students going abroad have always resorted, even after the great increase of the last few years, is still lower than in America, and in particular, the tuition fees are much less than in many of the larger American institutions, especially of the East. These facts largely counterbalance the expense of the trip across the ocean. Finally, it is to be remembered that a year or two of mathematical study in Germany, France, or Italy gives the student a reading and speaking knowledge of one of the great languages of modern thought, besides his own native English, such as can hardly be acquired in any other way.

When we come to mathematical considerations, the first question we must ask is whether getting a degree or learning mathematics is the prime object of the student going abroad. It is the former which, owing to circumstances mentioned in § 4, is too often uppermost in his mind. A student of this category had much better go abroad for his degree than to a second-rate American institution. Of course some care must be exercised

abroad. Such information might be secured. We content ourselves with giving two such items, for which we are indebted to Dr. Dunham Jackson:

At Göttingen in the years 1889–1909, inclusive, 22 Americans received the degree in mathematics, while no degrees in mathematics had been conferred on Americans during the four previous years.

At Leipzig in the years 1885–1902, inclusive, 8 Americans received the degree in mathematics, while after this time Americans seem to have ceased taking the degree in mathematics there.

At present from 2 to 4 Americans take their degree in mathematics in Germany each year, as against an average of 16 or 17 in the United States.

* There are also cases in which he takes so kindly to foreign conditions as to become out of touch with America. It is, however, rare that this state of affairs should survive his return more than a few months.

by him in the choice of his university, or he must have good fortune in writing a thesis whose weak points are not evident on a superficial examination, but his task is, on the whole, not a difficult one, and he gets at least the advantage of a period of foreign residence.

For another class of men foreign study may be recommended without qualification, namely, for able students who have already had a substantial training in one of the better American graduate schools, or who have even taken the doctor's degree at such a school. Such men will naturally go either to one of the great mathematical centers like Paris or Göttingen, where they will have the opportunity to hear lectures by several of the leading mathematicians of the day, and, perhaps, to see some of them occasionally outside of the lecture room; or they will select some mathematician of eminence in a particular field with whom they may hope to gain direct personal contact, and go to the university where he happens to be. Thus of late years a small but steady stream of American students has gone to Italy.

To the students just considered, and to some extent to their weaker comrades mentioned above, the period of residence at a great European mathematical center or of contact with an eminent mathematician at a less important European institution brings with it a realization of what high scientific ideals in mathematics are, and to what an extent they prevail abroad. Such ideals prevail also, it is true, at the strongest American institutions; but it is hard for the young American to appreciate their great diffusion in a ripened civilization until he has experienced it by personal contact.

SHORTER NOTICES.

Shop Mathematics. By EDWARD E. HOLTON, Head of the Department of Machine-Shop Practice, the Technical High School, Springfield, Mass. Springfield, The Taylor-Holden Company, 1910. xi+211 pp.

THE present rapid development of secondary vocational schools and their competition with general secondary schools have set in operation forces which tend to modify materially the character of secondary mathematical teaching in this country.

Since the mathematical work of the higher institutions rests upon the foundation laid by the secondary schools, it is important that any tendency to alter that foundation should be fully appreciated.

The book under review is one of the first published results of the attempt to develop a course in mathematics adapted to the needs of the secondary technical school, and as an indication of those needs, and possibly of the character of the mathematical instruction in such schools, it deserves serious attention.

The pioneers of the movement of protest against the traditional curriculum and methods of the secondary school have set high standards for the work of their own institutions. Witness the statement of Dr. C. M. Woodward:* "No blow is struck by him" (the pupil), "no line drawn, no motion regulated by mechanical habit. The only habit acquired is that of thinking. The quality of his every act springs from the conscious will, accompanied by a definite act of judgment."

Let us test the book before us by this standard, remembering that its object is to teach the pupil "what the shop problems are"† and "how to apply mathematical principles, rules, and formulas to the solution of such problems" and that "no attempt is made to teach mathematical theory or principles."

Consider the following problem (Problem 17, page 28). "What length of bar will be required to raise a building of 100 tons weight with 10 lbs. pull each on 100, ¼ in. lead jack-screws?" Result (page 191), 7.957 inches. If the reader, or better, the author, will try a simple experiment with a spring balance he will find that a force of ten pounds can easily be exerted by merely bending the little finger. The length found for the bar is about an inch more than that of an ordinary penholder. Now just picture a hundred laborers at work raising a building, each pulling with the little finger of one hand on a bar an inch longer than a penholder. Then picture what happens to the "shop foreman of twelve years' experience" when the contractor who is paying the gang of one hundred laborers, comes around and sees what they are doing! Again, consider the computed result: seven and nine hundred fifty-seven thousandths inches. Remember this is a problem on the

* Report of the U. S. Commissioner of Education, 1893–4, part I, p. 896.

† Introduction, p. viii. This introduction is by Dr. C. F. Warnier, principal of the Springfield, Mass., Technical High School and of the Evening School of Trades.

jack-screw, and the power is applied, not by means of a knife edge, but by the hand. The author gives the length of the bar to the thousandth of an inch!

This is not an isolated but a typical example of the "definite act of judgment" which accompanies the author's consideration of the data and results of his problems. For example, in problem 12, page 51, the pupil is asked to find the weight which can be raised by a force of 60 lbs. applied to a somewhat complex combination of screw and gears. The answer is given (page 195) as 203,575.68 lbs., i. e., to one part in twenty million. If the given force of 60 lbs. be altered by an amount equal to the weight of a single drop of water the computed result will be affected by more than the 0.68 lbs. appearing in the answer. This sort of thing is not a result of mere inadvertence in writing down the answers; it appears in the illustrative examples which are intended to serve the pupil as models. For example, in a problem concerning the friction of a lathe (page 54), where the coefficient of friction is given as .08, we have the following work:

$$400 \times .08 = 32, \quad \frac{32 \times 1\frac{1}{2}}{12} = 4,$$

$$\frac{50 \times 2\pi r}{12} = 50 \times 6.2832 = 314.16,$$

$$\frac{314.16 \times 4}{33000} = \frac{1256.64}{33000} = .038.$$

The coefficient of friction is known here only to one part in 8 but π is used to five significant figures, i. e., to one part in thirty thousand.

The pupil accustomed to books "written almost entirely from the point of view of the teacher of pure mathematics with little reference to concrete problems of life and having no reference whatever to the actual problems of the drafting room and the shop"* may be pardoned if he asks for information concerning the concreteness and actuality of problems, results, and methods of which the above are samples. With such models before him is the pupil likely to acquire the habit of thinking?

The author seems not to have realized that the standards of efficiency governing the mathematical work of the shop are

* Cf. Introduction, p. vi.

entirely analogous to those by which the mechanical work is judged.

In a little Handbook for Apprenticed Machinists issued by a prominent firm of manufacturers primarily for the use of their own apprentices the following statement appears: "He" (the apprentice) "should know the parts that are to be accurate; if a part is machined only to make it smooth, there is no need to take time to size it accurately." Had the author caught the spirit of this and applied it to the construction of his book, he would not have given the length of a jack-screw bar to the thousandth of an inch or a weight of a hundred tons to the hundredth of a pound.

The author's mathematical methods exhibit the same lack of appreciation of the essential features of a problem as is displayed in his data and results. For example, a problem in spiral gearing (page 45) involves the following relations:

$$P_1 = \frac{\pi}{P}, \quad P = \frac{P_1}{\cos Y}, \quad N = \frac{\pi D}{P}.$$

The data are $P = 16$, $D = 1\frac{1}{2}$, $Y = 60°$; the required quantity, N. The author's solution is

$$P_1 = \frac{3.1416}{16} = .19635,$$

$$P = \frac{.19635}{\cos 60°} = \frac{.19635}{.5} = .3937.$$

Then

$$\text{No. of teeth (i. e., } N) = \frac{1\frac{1}{2} \times \pi}{.3937} = \frac{4.7124}{.3937} = 12.$$

Could a better example be found of how *not* "to apply mathematical . . . formulæ in the solution of such problems"?* In the three pages of explanation of the principles of spiral gearing which the author has prefixed to this solution it has not occurred to him to collect the formulas bearing upon it and to deduce the simple relation $N = DP \cos Y$, which gives at once, on inserting the data, $N = \frac{3}{2} \cdot 16 \cdot \frac{1}{2} = 12$.

This particular performance is especially provoking in that it violates the cardinal principles both of efficient and of accurate shop practice. Efficiency calls for the avoidance of un-

*Introduction, p. viii.

necessary operations, accuracy, the restriction as far as possible of the number of operations of adjustment, e. g., the centering of a piece of work in the lathe. In the example quoted the author has violated the first principle by executing processes which are equivalent to multiplication followed by division by π; he has violated the second in that both these operations are necessarily effected with approximate numerical values and are consequently unnecessary operations of adjustment.

The mathematical content of the book consists largely in the statement of formulas and the substitution of numerical data therein. As indicated in the problem on spiral gearing, the author does not make efficient use of even the small amount of elementary mathematical knowledge which may reasonably be supposed to be in the pupil's possession. The notation used is frequently made unnecessarily cumbersome by the use of two or more letters where one would be sufficient, e. g., Wa for weight arm (pages 8, 10), CP for circular pitch (page 33). This last notation seems a wholly unnecessary and undesirable complication. Not only do both C and P appear in the problem, but their product is not the circular pitch which the author calls CP. Moreover the notation used in practice, in so far as the reviewer can judge from the publications of the manufacturers, and from a widely used Mechanical Engineers' Pocket-Book, to both of which the author acknowledges obligation, is not that adopted by him.

The introduction contains the following statement* "What is needed is to purge the old books of useless material and put in place of it practical mathematical work distinctly planned to make up for the shortcomings of the old methods when measured by the practical demands of modern times." It would be impossible to state more accurately the process to which this book should be subjected.

The vocational schools have already attained a prominent position in our system of education. For the interest of these schools themselves as well as for that of the other secondary and the higher schools it is important that their mathematical instruction shall be of a high standard. As to the precise content of the curriculum there is room for much honest difference of opinion. When however we consider that the problems of the shop demand besides the elementary operations such things as geometric progression (cone pulleys and hoisting

* Introduction, p. viii.

tackle), cube root (belting problems), Euclid's algorithm of the greatest common divisor and continued fractions* (gearing and screw problems), it would appear that if the course in mathematics is to do nothing more than provide a secure foundation for the work in the shop its abstract content is not likely to be much less extensive than has been usual in secondary schools. The nature of the concrete problems will naturally vary with the requirements of the different schools. But whatever their nature they must be *real* shop problems, that is they must be such that in solving them the pupil is compelled to consider not only the purely mathematical element but also the significance and reasonableness of the numerical data and results and the appropriateness of the algebraic and arithmetic processes used in their solution. Excessive formalism has been the bane of the teaching of abstract mathematics. It is just as common and just as pernicious in the shop as in the class-room.

CHARLES N. HASKINS.

Die Mathematik in den physikalischen Lehrbüchern. Von H. E. TIMERDING. Band III., Heft 2. Leipzig, Teubner, 1910. vi + 112 pp.

IN the systematic study of the teaching of mathematics in Germany which is being made under the auspices of the International commission of the teaching of mathematics, the present volume covers the field of the mathematics required and used in the physics of the "Höhere Lehranstalten" and "Hochschulen." The principles of mathematics found in the text-books on physics in use in Germany to-day form the basis of the discussion.

While the author states that the mathematics of physics is mainly of a geometric nature, it is easily seen that he considers the main problem to be concerned with the amount and quality of the infinitesimal analysis used in the texts investigated. Fundamentally, the problem may be stated as follows: The exact theory of most physical phenomena in its development requires a use of the principles of infinitesimal analysis. A scientific attitude toward these problems on the part of the instructor will not allow him to be satisfied with mere formulas or even a confused word picture, or the skillful manipulation of a "near calculus" which may interest but not convince the

* Practical Treatise on Gearing. Brown & Sharpe Mfg. Co., pp. 130–134.

student. The instructor in physics may not count on a thorough training on the part of all his students in the rigid proofs of these fundamental principles. What ought to be his attitude concerning the use of these principles?

As illustrations we present the cases of falling bodies and moment of inertia. They are both included in a course in physics. The former requires either two differentiations or two integrations; while the exact theory of the latter requires fundamentally the principles of integration. The former is usually handled by the aid of the notions of average velocities; while, in the case of the latter, in the expression $\sum mr^2$ the size of the m is — in some cases — carried down to the molecule. Most texts do not go to the limit. Hence the author's statement that the important thing is not what is said but what is left unsaid and that the instructor, if honest, must clearly recognize the necessity of the principles of the infinitesimal analysis. The statement is further made that one reason why so many subterfuges, instead of the symbols of the calculus, are used is because the authors wish their books to sell. The notions of the infinitesimal analysis would not confuse the subject but clear it up. Calculus symbols should not be used as loosely as they are. It is an evidence not of good pedagogy but rather of downright deceit not to use them.

An excellent chapter on the history of physics and its problems, from the beginning of the seventeenth century when it was a branch of the Aristotelian philosophy, to the present with its wondrous development, is included. The correlation of mathematics and physics is discussed, the influence of physics in enriching the mathematics and of the latter in putting physics on a safe foundation of rigorous proofs.

It is pointed out that Germany has lagged behind both England and the United States in the use of such aids as graphs, accurate drawings and illustrations, and others suggested by technology, which help to give the student clear and definite notions of mathematical or physical problems.

The problem of the requisite instructions in mathematics presented from the viewpoint of the teacher of physics is clearly stated and a scientific attitude is called for. What the nature of the correlation of instruction should be the author but slightly suggests. It would seem to call for a training at an earlier period in the principles of the infinitesimal analysis. This correlation is being worked out today.

ERNEST W. PONZER.

Practical Measurements. By A. W. SIDDONS and A. VASSALL. Cambridge, The University Press, 1910. xiv + 60 pp.

THE definite purpose of this little note-book is to furnish for the boys at Harrow a series of elementary experiments in measuring and weighing, conforming with the recommendations embodied in the recent report on "The correlation of mathematical and science teaching" drawn up by a joint committee of the mathematics and science teachers of England.

On the mathematical side the experiments deal with the measurements, correct to three significant figures, and the necessary computations — of lengths, areas, and volumes — a practical checking up on the theorems of geometry.

The experiments emphasizing the side of physics deal with the weighing of liquids and solids and the determination of their specific gravities. Experiments with the hydrometer, barometer, pendulum, the U tube, and on Boyle's law are included. In the appendix experiments with the vernier, micrometer screw-gauge, and spherometer are added.

The methods used throughout the book are its strongest feature. Boys thoroughly grounded in the fundamental principles emphasized throughout will be able to do their future work with greater efficiency. Accuracy in estimating, measuring, computing, laying out to scale, and orderly arrangement are required. Concrete results, which should check up with theory, are required throughout. The practice of requiring such definite results, correct to a certain number of significant figures, is well worth while.

ERNEST W. PONZER.

Azimuth. By G. L. HOSMER. New York, John Wiley and Sons, 1909. v + 73 pp.

IN ordinary engineering practice the determination of the azimuth of some line of a survey by astronomical means is not necessary; and in the cases where it is necessary, or at least advisable, the determination is likely to be a serious matter for many a practical surveyor—not because of any inherent difficulty in making or reducing the observations with all the accuracy that is needed in the field, but because the available methods are scattered in books which stay at home. Hence the utility of Hosmer's Azimuth, wherein are collected and explained with all simplicity various practicable field methods of fixing the true bearing of a line to minutes (or even to within

a few seconds). In so short space may ample descriptive matter and illustrated examples be put that only about half the book, small as it is, is used for the purpose; the other half consists of the requisite astronomical, physical, and mathematical tables. The surveyor who is no astronomer need have no fear. The author requires of him no knowledge of theoretical or practical astronomy or of uranography—except that the totally ignorant might have difficulty in picking out Markab (α Pegasi) from the chart on which there are two α's in the great square of Pegasus.

E. B. WILSON.

EISENHART'S DIFFERENTIAL GEOMETRY.

IN reviewing Professor L. P. Eisenhart's book on Differential Geometry in the June number of the BULLETIN, I criticized four equations which occur at the bottom of page 313 without noticing that the arc is supposed to be the parameter along the locus of the centers of the spheres, as is stated in the last paragraph of page 310. The equations are correct as they stand with the exception of a plus sign between the parentheses in the third, which should be minus.

Professor Eisenhart has sent me the following list of errata, the publication of which may be of convenience to readers of the book:

P. 51, Ex. 24 is incorrect.

P. 87, omit the last line.

P. 117, to the expression for $\frac{d^2x}{ds^2}$ add $\frac{\partial x}{\partial u}\frac{d^2u}{ds^2} + \frac{\partial x}{\partial v}\frac{d^2v}{ds^2}$.

P. 123, last paragraph is incorrect. The position of the surface relative to the tangent plane depends upon the character of the terms of third and higher orders.

P. 179, in (86) the signs + should be −, and in the next equation the signs − should be +.

P. 180, line 27, change S to S_2.

P. 221, line 24, the upper limit of the second integral should be u_2 instead of u.

P. 241, in (52) change y_1 to y_0'.

P. 259, line 3, in place of "congruent" read "superposable by a translation."

P. 268, Ex. 7, change the last clause to "determine the two lines of striction."

P. 278, line 8, insert after "equations" the phrase "and §§ 77, 82."

P. 280, line 10, after "values" insert "(cf. §§ 77, 82)."

P. 313, in the expression for ζ change $+$ before $(u^2 - \frac{1}{2})$ to $-$.

P. 400, line 9, change ω_1, ω_2 to $\omega_1 + \frac{\pi}{2}, \omega_2 + \frac{\pi}{2}$.

P. 412, Ex. 6, change $+$ before ds_1 to $-$.

P. 418, line 19, remove the sign Σ after m^2.

P. 423, Ex. 9, in the equation change λ to λ^2.

P. 441, line 27, after "zero" insert "in fact L vanishes identically."

P. 444, Ex. 15, in last term of the equation change $+$ to $-$ and p to p_1.

G. A. BLISS.

NOTE ON COLLINEATION GROUPS.

SINCE the publication of my paper on collineation groups in the *Transactions* (volume 12, No. 2, April, 1911) my attention has been called to a similarity which exists between my determination of the collineation groups in the ordinary plane and that given by Valentiner ("De endelige transformations-gruppers theori," *Videnskabsselskabets Skrifter*, 6 Raekka, Copenhagen, 1889).

The general outlines of the first parts of the two papers are the same, as in both the groups which contain homologies of higher period than 2 are first discussed. A determination of those groups leaving a line invariant which must contain an homology of period 2 having that line for axis is given in both papers. The proofs that no group can contain homologies of higher period than 5 or homologies of period 5 are essentially the same. The proofs that no group can contain homologies of period 4 are somewhat different. Valentiner's discussion of groups containing homologies of period 3 is inaccurate and as a consequence he overlooks the existence of the G_{216}, although he considers the possibility of a group of this order.

The rest of the discussion is in the main quite different. I proved a theorem concerning the order of abelian groups contained by non-abelian groups of three times their order which is similar to his, but made no use of it in the determination of the groups in the ordinary plane. He arrived at the G_{168} and the G_{360} as the result of the solution of a diophantine equation (the same, in fact, as was used by Jordan in his attack on the problem, *Crelle*, 1878), whereas I arrived at them from the consideration of groups which left conics invariant. I however made use of a special type of this equation in two or three places.

H. H. MITCHELL.

NOTES.

THE annual meeting of the AMERICAN MATHEMATICAL SOCIETY will be held in New York on December 27–28. The winter meeting of the Chicago Section will be held at the University of Chicago on December 29–30. Titles and abstracts of papers to be presented at these meetings should be in the hands of the respective secretaries by December 9. Abstracts intended to be printed in advance of the meeting should be sent in as early as possible.

THE Annual Register of the Society is now in preparation and will be issued in January. Blanks for furnishing necessary information have been sent to the members. Early notice of any changes since the issue of the last Register will greatly facilitate the work of the Secretary. The Register is widely circulated and it is desirable that the information which it contains should be accurate and reliable.

THE concluding (October) number of volume 12 of the *Transactions of the American Mathematical Society* contains the following papers: "On the limit of the degree of primitive groups," by W. A. MANNING; "Isomorphisms of a group whose order is a power of a prime," by G. A. MILLER; "On minimal lines and congruences in four-dimensional space," by J. EIESLAND; "Volterra's integral equation of the second kind, with discontinuous kernel. Second paper," by G. C. EVANS; "One-parameter families and nets of plane curves," by E. J. WILCZYNSKI; Notes and errata, volumes 10 and 11.

THE opening (September) number of volume 13 of the *Annals of Mathematics* contains: "A method of solving linear differential equations." Second paper, by P. A. LAMBERT; "Duality in projective geometry," by N. J. LENNES; "A fundamental parametric representation of space curves," by L. P. EISENHART; "Generalization in the theory of numbers and theory of linear groups," by MILDRED SANDERSON; "Transformations of series by means of functions admitting a recurrent relation," by W. C. BRENKE; "A theorem on (m, n) correspondences," by L. I. NEIKIRK. With the present volume, the *Annals* appears in a new dress, with narrower margins and a change in style of type. The editors are Ormond Stone, Maxime Bôcher, G. D. Birkhoff, L. P. Eisenhart, Elijah Swift, Oswald Veblen, and J. H. Maclagan-Wedderburn.

THE first number has appeared of the *Tôhoku Mathematical Journal*, edited by Professor T. HAYASHI of the Imperial University, Sendai, Japan, with the cooperation of M. Fujiwara, F. Ishiwara, T. Kubota, and K. Ogura.

THE Italian society for the advancement of science held its annual session in Rome during the week October 12–18. As this is the fiftieth year of Italian independence, it seemed appropriate to give a summary of the progress made in science by Italians during the last fifty years. The following reports were made in pure and applied mathematics: "Astronomy," by E. BIANCHI; "Geodesy," by V. REINA; "Theory of functions and integral equations," by G. LAURICELLA; "Geometry," by U. AMALDI; "Mathematical physics," by T. LEVI-CIVITA; "Electricity," by M. O. CORBINO. The following general lectures were also delivered: "Evolution of the measurement of space and time," by G. CASTELNUOVO; "The new physics," by A. RIGHI; "Science in the far east," by G. VACCA; "On algebra of logic and some transcendants," by G. ANDREOLI; "Historical and critical note on the principles of mechanics," by R. GIACOMELLI.

THE sum of 20,000 rupees (nearly $7000.00) was recently appropriated by the University of Calcutta for the purpose of publishing certain ancient Indian mathematical manuscripts with an English translation.

GINN and Company announce the early appearance of a work on Advanced Calculus, by Professor E. B. WILSON, of the Massachusetts Institute of Technology.

THE reports of the American committees and subcommittees of the International commission on the teaching of mathematics are being published by the U. S. Bureau of Education, from which copies may be obtained gratis on application. The following reports have already appeared: Graduate work in mathematics in universities and in other institutions of like grade in the United States. Undergraduate work in mathematics in colleges of liberal arts and universities. Examinations in mathematics other than those set by the teacher for his own classes. Mathematics in the technological schools of collegiate grade in the United States. Training of teachers of elementary and secondary mathematics.

UNIVERSITY OF PARIS. The following mathematical courses are announced for the semester beginning November 3, 1911:—By Professor G. DARBOUX: Differential geometry, quadratic differential forms, applications of the calculus of variations, two hours.—By Professor E. GOURSAT: Differential and integral calculus, elements of the theory of analytic functions, two hours.—By Professor E. BOREL: Elliptic functions, one hour.—By Professor P. APPELL and M. CL. GUICHARD: General laws of equilibrium and motion, two hours.—By MM. E. CARTAN and MONTEL: General mathematics, first part, two hours.—By Professor H. POINCARÉ: Theory of newtonian potential and its applications to the figure of planets, two hours.—By Professor J. BOUSSINESQ: Mathematical theory of elasticity of solid bodies, two hours.—By M. A. CAHEN: Indeterminate analysis of the second degree with two unknowns, quadratic forms and algebraic numbers, two hours.

Conferences will be conducted by MM. LEBESGUE, GUICHARD, CARTAN, MONTEL, SERVANT, and ROUBAUDI.

In the Ecole Normale. By Professor E. VESSIOT: Differential and integral calculus.—By Professor E. BOREL: Theory of functions.

In the second semester, beginning March 1, the following courses are announced:—By Professor E. PICARD: Fourier's series and its generalizations.—By Professor E. GOURSAT: Ordinary and partial differential equations.—By Professor P. APPELL: General laws of motion of systems, analytic mechanics, hydrostatics and hydrodynamics.—By M. E. CARTAN: Analysis and mechanics.—By ———: Indeterminate analysis of the second degree.

In the Ecole Normale the courses of Professors Vessiot and Borel are continued from the first semester and there is added: By Professor J. HADAMARD: Mathematics.

THE following courses in mathematics are announced at the German technical schools (and at Delft, Holland) during the present winter semester. The course designated as "higher mathematics" refers to a combination of algebra, analytic geometry, differential and integral calculus, continuing for from two to four semesters I, II, etc.

AACHEN.—By Professor O. BLUMENTHAL: Higher mathematics, II, six hours; Theory of aviation, four hours.—By Professor E. KÖTTER: Descriptive geometry, six hours; Graphical statics, three hours.—By Professor K. REISSNER: Mechanics, I and II, four hours; Aerodynamics, four hours.

BERLIN.—By Professor G. HETTNER: Differential and integral calculus, six hours; with exercises, two hours; Differential equations, two hours.—By Professor F. HAENTZSCHEL: Elements of the calculus, with exercises, six hours; Elements of mechanics, four hours.—By Professor O. KRIGAR-MENZEL: General mechanics, four hours; Advanced mechanics, four hours.—By Professor E. JOLLES: Descriptive geometry, I, six hours; Descriptive geometry, II, six hours.—By Professor E. LAMPE: Calculus, with exercises, six hours.; Definite integrals, four hours.—By Professor G. SCHEFFERS: Descriptive geometry, I and II, each six hours.—By Professor G. WALLENBERG: Selected chapters of elementary mathematics, two hours; Theory of functions, two hours; Theory of potential, two hours.—By Professor E. SALKOWSKY: Descriptive geometry, six hours.—By Dr. L. LICHTENSTEIN: Integral equations, six hours.—By Dr. R. FUCHS: Differential equations, two hours.—By Dr. K. v. IGNATOWSKY: Vector analysis, two hours.

BRESLAU.—By Professor C. CARATHÉODORY: Higher mathematics, four hours; Selected chapters in analysis, two hours; Mathematics for chemists and mining engineers, three hours.—By Professor G. HESSENBERG: Descriptive geometry, with exercises, eight hours; Selected chapters in geometry, three hours.—By Professor E. STEINITZ: Higher mathematics, III, with exercises, eight hours.

BRUNSWICK.—By Professor R. DEDEKIND: Elements of the theory of numbers, four hours; Introduction to the theory of

probabilities.—By Professor R. FRICKE: Analytic geometry and algebra, five hours; Differential and integral calculus, four hours; Selected chapters in the theory of functions, three hours. —By Professor H. E. TIMERDING: Descriptive geometry, six hours; Introduction to higher analysis, four hours; Theory of surfaces and space curves, three hours.—By Dr. K. WITTE: Analytic mechanics, four hours.

DANZIG.—By Professor H. LORENZ: Dynamics of a rigid body, four hours: Elasticity, two hours.—By Professor H. v. MANGOLDT: Higher mathematics, five hours; Seminar, two hours.—By Professor F. SCHILLING: Descriptive geometry, with exercises, eight hours; Perspective and photogrammetry, one hour.—By Professor J. SOMMER: Higher mathematics, six hours; Seminar (with Professor v. Mangoldt) two hours.

DARMSTADT.—By Professor F. DINGELDEY: Higher mathematics, six hours; Theory of attraction, two hours.—By Professor F. GRAEFE: Repetitorium of elementary mathematics, three hours; Higher mathematics, five hours; History of mathematics, one hour.—By Professor J. HORN: Higher mathematics, I and II, each five hours.—By Professor R. MÜLLER: Descriptive geometry, eight hours.—By Professor H. WIENER: Descriptive geometry, eight hours; Foundations of elementary geometry, two hours.

DELFT.—By Professor J. A. BARRAU: Determinants and introduction to the calculus, three hours; Elements of projection two hours; Curved lines and surfaces, two hours; Line geometry, one hour.—By Professor W. H. L. JANSSEN v. RAAY: Advanced analytics and foundations of the calculus, five hours; Advanced calculus, four hours.—By Professor W. A. VERSLUYS: Plane analytic geometry, two hours; Geometry of space, two hours; Theory of equilibrium and of motion, two hours.—By Professor J. CARDINAAL: Descriptive geometry, two hours; Application to space curves and surfaces, two hours; Kinematics, three hours.—By Dr. G. SCHOUTEN: Geometry of motion, two hours; Moments of inertia and dynamics of a particle, two hours; Theory of the gyroscope, one hour.

DRESDEN.—By Professor G. HELM: Higher mathematics, five hours; Mathematical physics, three hours; Theory of the potential, three hours.—By Professor M. KRAUSE: Higher mathematics, five hours; Analytic functions, three hours;

Seminar, two hours.—By Professor L. LUDWIG: Descriptive geometry, six hours; Theory of perspective, two hours; Space curves, two hours.—By Professor E. NAETSCH: Analytic geometry of quadric surfaces, four hours; Elementary algebra and analysis, two hours.—By Dr. L. HEGER: Plane cubic curves, three hours.

HANOVER.—By Professor L. KIEPERT: Higher mathematics, I, eight hours; Higher mathematics, III, two hours; Calculus of variations, two hours.—By Professor F. MÜLLER: Higher mathematics, II, eight hours; Vector analysis, two hours.—By Professor K. WIEGHARDT: Foundations of higher mathematics, four hours; Elasticity, four hours; Differential equations, one hour.—By Professor K. RODENBURG: Descriptive geometry, nine hours.—By Dr. L. PETZOLD: Algebra and trigonometry, three hours.

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STUTTGART.—By Professor R. MEHMKE: Descriptive geometry, six hours; Graphical methods, three hours.—By Professor L. KOMMERELL: Foundations of geometry, four hours.—By Professor E. WÖLFFING: Theory of functions, I, four hours.—By Professor G. FABER: Higher mathematics, II, six hours; Differential and integral calculus, with exercises, four hours.—By Dr. F. HALLER: Plane and spherical trigonometry, four hours.

PROFESSOR E. W. BROWN, of Yale University, has been elected an honorary fellow of Christ's College, Cambridge, England.

PROFESSOR F. LONDON, of the University of Bonn, has been promoted to a full professorship of mathematics.

PROFESSOR F. ENGEL, of the University of Greifswald, has been decorated with the order of the red eagle of the fourth class.

PROFESSOR E. STUDY, of the University of Bonn, has been elected to membership in the Göttingen academy of sciences.

PROFESSOR N. HATZIDAKIS, of the University of Athens, has been elected a member of the Vienna academy of sciences. Professor D. HILBERT, of the University of Göttingen, has been elected a corresponding member.

PROFESSOR PH. FURTWÄNGLER, of the agricultural academy of Poppelsdorf, has accepted a full professorship of mathematics at the University of Vienna.

DR. R. BÖHM has been appointed docent in mathematics at the University of Munich.

DR. J. L. JONES has been appointed instructor in mathematics at Yale University.

PROFESSOR L. G. WELD, until recently head of the department of mathematics and dean of the faculty of arts of Iowa State University, has been appointed director of the Pullman Free School of Manual Training to be established at Pullman, Ill.

PROFESSOR GEORGE W. JONES, of Cornell University, died October 29, at the age of 74 years. He retired as professor emeritus under the Carnegie foundation in 1907, after 45 years of teaching, the last 31 of which were spent at Cornell.

RECENT catalogues of second-hand books:—Heinrich Kerler, Ulm a. Donau, Antiquariats-Katalog 397, 2377 titles in pure and applied mathematics.—W. Heffer and Sons, Cambridge, England, catalogue 83, 2141 titles in mathematics, physics, and astronomy; also catalogue 81, 4507 titles of which 927 are school and college text-books in mathematics.—W. Junk, Kurfürstendamm 201, Berlin, bulletin 9, 2700 titles in exact sciences.—Ferdinand Schöningk, Osnabrück, catalogue 125, 1865 titles in mathematics and sciences.—J. Schweitzer, Lenbachplatz 1, Munich, catalogue 54, insurance, 2337 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BREGGEN (J. v. D.). Vademecum der Wiskunde. Zutphen, 1911. 8vo. 81 pp. M. 1.00

CAILLER (C.). Sur les notion de courbure et sur quelques points de géométrie infinitésimale non-euclidienne. Paris, Fischbacher, 1911. 4to. 60 pp. Fr. 5.00

CHAMBRÉ (A.). Darstellung von Faktoren ganzer Funktionen durch Kovarianten. (Diss.) Giessen, 1910. 8vo. 36 pp.

DAHLHAUS (G.) und RIEMANN (A.). Mathematisches Formelbuch. 2 Teile. Berlin, Günther, 1911. 107 pp. M. 4.00

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Edition française. Tome III, 3me volume: Géométrie algébrique plane. Fascicule 1: Les coniques, par F. Dingeldey et E. Fabry. Leipzig, Teubner, 1911. 8vo. pp. 1–160. M. 6.00

EULER (L.). Opera omnia. Sub auspiciis Societatis Scientiarum Naturalium Helvetiae edenda curaverunt F. Rudio, A. Krazer et P. Stäckel. (45 Bände in 3 Serien.) Serie I, Band 1: Vollständige Anleitung zur Algebra, mit den Zusätzen von J. L. Lagrange. Herausgegeben von H. Weber, mit Vorwort zur Euler-Ausgabe und der Lobrede von N. Fuss. Leipzig, 1911. 4to. M. 28.50

FINZEL (A.). Die Lehre vom Flächeninhalt in der allgemeinen Geometrie. (Diss.) Strassburg, 1911. 8vo. 46 pp.

GAMBIER (G.). Le mathématicien François Viète. Généalogie de sa famille. La Rochelle, Texier, 1911. 8vo. 31 pp.

INTERNATIONAL catalogue of scientific literature. Ninth annual issue. A: Mathematics. London, Harrison, 1911. 8vo. 228 pp. 15s.

KIEFER (A.). Die Einführung der homogenen Koordinaten durch K. W. Feuerbach. (Diss.) Strassburg, 1910. 8vo. 55 pp.

KILL (P.). Beiträge zum Fundamentalproblem der Flächentheorie. (Diss.) Strassburg, 1910. 8vo. 49 pp.

LUKJANTSCHENKO (S.). Integrirung der Differentialgleichungen. Mit Anhang: Doppelte und dreifache Integrale. 2te Auflage. (Russian.) Charkow, 1911. 8vo. 226+51 pp.

MANGOLDT (H.). Einführung in die höhere Mathematik für Studierende und zum Selbststudium. 1ter Band: Anfangsgründe der Infinitesimalrechnung und der analytischen Geometrie. Leipzig, Hirzel, 1911. 8vo. 14+477 pp. M. 13.00

MACMAHON (P. A.). Memoir on the theory of the partition of numbers. Part 6. (Royal Society.) London, Dulau, 1911. 4to. Sewed. 1s. 6d.

MATHEMATICAL QUESTIONS and solutions from the Educational Times, new series. Volume 19. London, Hodgson, 1911. 8vo. 6s. 6d.

MILNE (W. P.). Projective geometry for use in colleges and schools. London, Macmillan, 1911. 8vo. 164 pp. 2s. 6d.

MUIR (T.). The theory of determinants in the historical order of development. Volume 2: The period 1841 to 1860. London, Macmillan, 1911. 8vo. 492 pp. 17s.

PICKERT (E.). Verallgemeinerung der Untersuchungen von Gauss über das arithmetisch-geometrische Mittel. (Diss.) Leipzig, 1911. 8vo. 67 pp.

RIEMANN (A.). See DAHLHAUS (G.).

SALTYKOV. Sur l'évolution de la théorie des équations partielles du premier ordre d'une fonction inconnue. (Russian.) St. Petersburg, 1911. 8vo.

SEEMAN (H.). Projektive Verallgemeinerung metrischer Begriffe. (Diss.) Giessen, 1910. 8vo. 24 pp.

THAER (F.). Analytische Beiträge zur Lehre vom Kegelschnittsystem (3 p, 1 l). (Diss.) Giessen, 1911. 28 pp.

VAERTING (M.). Zur Transformation der vielfachen Integrale. (Diss.) Giessen, 1910. 8vo. 24 pp.

WOLFF (G.). Ueber Kollineationen in der Ebene. (Diss.) Giessen, 1910. 8vo. 60 pp.

ZITSCHER (H.). Philosophische Untersuchungen über die Zahl. (Diss.) Greifswald, 1910. 8vo. 80 pp.

II. ELEMENTARY MATHEMATICS.

BÉCHÉ (A.). See DUSSAUX (E.).

BIOCHE (C.). Enseignement secondaire. (Rapports de la sous-commission française, volume II.) Paris, Hachette, 1911. 8vo. 157 pp. Fr. 4.00

BORCHARDT (W. G.) and PERROTT (A. D.). Geometry for schools. Volume 2. Stage 3. Section 1. London, Bell, 1911. 8vo. 118 pp. 1s. 6d.

BÖTTGER (A.). See THAER (A.).

BRANDENBERGER (K.). Der mathematische Unterricht an den Schweizerischen Gymnasien und Realschulen. Genève, Georg, 1911. 8vo. 163 pp. Fr. 3.50

BRYAN (G. H.) and DEAKIN (R.). The text-book of algebra. Part II: Senior. London, Clive, 1911. 8vo. 356 pp. 3s.

DEAKIN (R.). A text-book of algebra. Part I: Junior. London, Clive, 1911. 8vo. 372 pp. 3s.

——. See Bryan (G. H.).

DUSSAUX (E.) et BÉCHÉ (A.). Nouveau cours de géométrie. 3e année d'écoles primaires supérieures; préparation aux écoles nationales d'arts et métiers. Paris, Colin, 1911. 16mo. 391 pp. Fr. 3.00

DYCK (W. v.). Enseignement des sciences mathématiques, naturelles et techniques dans les écoles supérieures. Ixelles-Bruxelles, Bothy, 1911. 8vo. 67 pp.

GEUTHER (N.). See THAER (A.).

GROSSMANN (M.). Der mathematische Unterricht an der Eidgenössischen Technischen Hochschule. Genève, Georg, 1911. 52 pp. Fr. 2.00

HUN (J. G.) and MACINNES (C. R.). Logarithmic, trigonometric, and other tables. New York, Macmillan, 1911. 8vo. 2+104 pp. $0.60

——. The elements of plane and spherical trigonometry; with tables. New York, Macmillan, 1911. 8vo. 7+205 pp. $1.35

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INTERMEDIATE mathematics papers, from 1900 to 1911. (University Tutorial Series.) London, Clive, 1911. 8vo. 100 pp. Sewed. 2s.

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LIETZMANN (W.). See SCHUSTER (M.).

LÜBSEN (H. B.). Ausführliches Lehrbuch der Arithmetik und Algebra zum Selbstunterricht und mit Rücksicht auf die Zwecke des praktischen Lebens. 27te Auflage. Neubearbeitet von A. Dornadt. Leipzig, Brandstetter, 1911. 8vo. 4+283 pp. Cloth. M. 4.50

MACINNES (C. R.). See HUN (J. G.).

MAHLER (G.). Leitfaden für den Anfangsunterricht in der Algebra. 2te Auflage. Stuttgart, Bonz, 1911. 8vo. 6+127 pp. M. 1.50

MARSH (C. A.) and PHIPPS (H. J.). College entrance examination papers in plane geometry. New York, Merrill, 1911. 12mo. 178 pp. $0.60

PERROTT (A.). See BORCHARDT (W. G.).

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ZUSCHLAG (H.). Planimetrie. Für die Schüler der Quarta bis Oberprima und für Abiturienten. 2te Auflage. (Bibliothek Schüler-Versetzung. 7ter Band.) Berlin-Schöneberg, Mentor-Verlag, 1911. 112 pp. M. 1.00

——. Planimetrische Konstruktionsaufgaben (I). 2te Auflage. (Bibliothek Schüler-Versetzung. 8ter Band.) Berlin-Schöneberg, Mentor-Verlag, 1911. 69 pp. M. 1.00

——. Stereometrie. Anfängerkursus. 2te Auflage. (Bibliothek Schüler-Versetzung. 18ter Band.) Berlin-Schöneberg, Mentor-Verlag, 1911. 12+64 pp. M. 1.00

III. APPLIED MATHEMATICS.

BARTON (E. H.). Analytical mechanics, comprising kinetics and statics of solids and fluids. London, Longmans, 1911. 8vo. 556 pp. 10s. 6d.

BEDELL (F.) and PIERCE (C. A.). Direct and alternating current manual; with directions for testing, and a discussion of the theory of electrical apparatus. 2nd edition, enlarged. New York, Van Nostrand, 1911. 8vo. 13+360 pp. $2.00

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FERGUSON (O. J.). The elements of electrical transmission; a text-book for colleges and technical schools. New York, Macmillan, 1911. 8vo. 7+457 pp. $3.50.

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LANDRÉ (C. L.). Mathematisch-technische Kapitel zur Lebensversicherung. 4te Auflage, verbessert und vermehrt von H. F. Landré. Jena, 1911. 8vo. 26+528 pp. M. 11.50

MARTIN (L.) Text-book of mechanics. Volume 3: Mechanics of materials New York, Wiley, 1911 12mo 229 pp. $1.50

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SMOLIK-HELLER. Raumlehre und darstellende Geometrie fur die 4te bis 7te Klasse der Realschulen. Bearbeitet von K. Hahndel. 4te Auflage. Wien, Tempsky, 1911. K. 3.60

SPOFFORD (C. M.). Theory of structures. New York, McGraw-Hill, 1911. 8vo. 432 pp. $4.00

SSUSSLOW (G. K.). Elements of analytical mechanics. (Russian.) Volume 1, Part 2: Dynamics of a particle. 2nd edition. Kiew, 1911. 8vo. 155 pp. $0.75

THOMPSON (S. P.). Traité théorique et pratique des machines dynamo-électriques. Traduit par E. Boistel. Courant continu. 4e édition française. Paris, Béranger, 1911. 8vo. 18+1060 pp.

TIMBIE (W. H.). Elements of electricity for technical students. New York, Wiley, 1911. 12mo. 11+556 pp. Cloth. $2.00

VITALE (A.). Trattato elementare di geometria descrittiva. Avellino, Pergola, 1911. 8vo. 65 pp.

VOGT (H.). Geometrie und Ökonomie der Bienenzelle. (Festschrift zur Jahrhundertfeier der Universität Breslau.) Breslau, Trewendt & Granier, 1911. 8vo. 63 pp. M. 3.00

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ZÜHLKE (P.). Der Unterricht im Linearzeichnen und in der darstellenden Geometrie an den deutschen Realanstalten. (Abhandlungen über den mathematischen Unterricht in Deutschland. 3ter Band, 3tes Heft.) Leipzig, Teubner, 1911. 8vo. 4+92 pp. M. 2.60